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Valuation of Interest Rate Swaps Under a Discrete-Time No-Arbitrage Framework: Theory, Derivations, and Numerical Applications

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Abstract

This paper presents a rigorous treatment of interest rate swap valuation within a discrete-time no-arbitrage framework. We derive closed-form expressions for the par swap rate under both general (time-varying notional) and level-notional conditions, establishing the equivalence between the two approaches through the telescoping property of forward rates and discount factors. We extend the standard framework to deferred-start swaps and derive a corresponding simplified formula. All theoretical results are accompanied by detailed numerical examples that demonstrate practical implementation. A sensitivity analysis illustrates how the term structure of interest rates affects swap pricing, providing both pedagogical and practical insights for risk management. Our unified treatment bridges the gap between introductory textbook presentations and the measure-theoretic continuous-time models used in advanced quantitative finance.

Keywords: Interest rate swaps; no-arbitrage theory; discrete-time models; forward rates; financial mathematics.

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1.0 Introduction

Interest rate swaps are among the most widely traded financial derivatives in global markets. According to the Bank for International Settlements (BIS), the notional outstanding amount of interest rate swaps exceeded $400 trillion as of 2023 Bank for International Settlements (2023). An interest rate swap is a contractual agreement between two counterparties to exchange a stream of interest payments: one party pays a fixed rate while the other pays a floating (variable) rate, both applied to a specified notional principal over a predetermined period.

The economic motivation for interest rate swaps is multifaceted. Borrowers with variable-rate liabilities may seek to hedge against rising interest rates by entering into a swap to effectively convert their exposure to a fixed rate. Conversely, institutions with fixed-rate obligations may prefer variable exposure to better match floating-rate asset income. In either case, the swap provides a mechanism for transferring interest rate risk between parties with different preferences or comparative advantages in different markets—an observation first formalized by Bicksler et al. (1986).

The theoretical foundations of swap pricing rest on the principle of no-arbitrage, which requires that the present value of fixed-leg payments equal the present value of floating-leg payments at inception. This fundamental condition, combined with the term structure of interest rates, yields deterministic closed-form expressions for the par swap rate. While the continuous-time measure-theoretic treatment of swap valuation using the LIBOR market model Brace et al. (1997) and Miltersen et al. (1997) and the Heath–Jarrow–Morton (HJM) framework Heath et al. (1992) provides the standard for advanced quantitative applications, the discrete-time deterministic framework remains indispensable for actuarial practice, pedagogical purposes, and situations where the term structure is directly observable through zero-coupon bond prices.

The contribution of this paper is threefold. First, we provide a rigorous and self-contained derivation of the par swap rate formula under a general framework allowing time-varying notional amounts, and show how it simplifies under the common assumption of a level notional. Second, we extend the framework to deferred-start swaps and derive the corresponding closed-form expression, which is often omitted in introductory treatments. Third, we complement the theory with detailed numerical examples and a sensitivity analysis that illustrates the relationship between the shape of the yield curve and the resulting swap rate.

The remainder of this paper is organized as follows. Section 2 establishes the mathematical preliminaries, including precise definitions of spot rates, discount factors, and forward rates. Section 3 derives the general par swap rate formula. Section 4 presents the simplified formula for level notional amounts and proves its equivalence to the general formula. Section 5 extends the framework to deferred-start swaps. Section 6 provides comprehensive numerical examples. Section 7 presents a sensitivity analysis. Section 8 concludes and discusses extensions to continuous-time models.

2.0 Mathematical Preliminaries

We work in a discrete-time setting with a finite horizon $T \in \mathbb{N}$. Let $t_0 = 0 < t_1 < t_2 < \cdots < t_n = T$ denote the payment dates of the swap, where for simplicity we assume annual spacing so that $t_i = i$ for $i = 0,1,\ldots,n$.

Definition 1 (Spot Interest Rate). The spot interest rate r_t is the annual effective yield on a zero-coupon bond maturing at time t. Equivalently, r_t is the rate such that the price at time 0 of a zero-coupon bond paying \$1 at time t is $P_t = (1 + r_t)^{-t}$.

Remark 2. The collection $\{r_t\}_{t=1}^{n}$ constitutes the term structure of interest rates (or yield curve) observed at time 0. The corresponding discount factors $\{P_t\}_{t=1}^{n}$ are the prices of zero-coupon bonds and form an equivalent representation of the term structure.

Definition 3 (Forward Interest Rate). The periodic effective forward interest rate between times t_1 and t_2 (with $0 \le t_1 < t_2$), denoted $f^*_{[t_1,t_2]}$, is defined by $f^*_{[t_1,t_2]} = \frac{(1+r_{t_2})^{t_2}}{(1+r_{t_1})^{t_1}} - 1 = \frac{P_{t_1}}{P_{t_2}} - 1$. For consecutive annual periods, the one-period forward rate between t and $t+1$ is $f_{[t,t+1]} = \frac{(1+r_{t+1})^{t+1}}{(1+r_t)^t} - 1$.

The forward rate $f^*_{[t_1,t_2]}$ can be interpreted as the "break-even" rate for the period $[t_1, t_2]$: it is the rate that makes an investor indifferent between (i) investing for t_2 years at rate r_{t_2} and (ii) investing for t_1 years at rate r_{t_1} and then reinvesting at the forward rate for the remaining $t_2 - t_1$ years.

Lemma 4 (Telescoping Property). For any $t \in \{1,2,\ldots,n\}$, $f^*_{[t-1,t]} \cdot P_t = P_{t-1} - P_t$.

Proof. By Definition Definition 2.2,

$$f^*_{[t-1,t]} = \frac{P_{t-1}}{P_t} - 1,$$

so that

$$f^*_{[t-1,t]} \cdot P_t = \left(\frac{P_{t-1}}{P_t} - 1\right) P_t = P_{t-1} - P_t.$$

This identity is crucial for deriving the simplified swap rate formula in Section 4.

3.0 The General Par Swap Rate

Consider an interest rate swap with n settlement periods, where the notional amount at time t_i is Q_{t_i} (which may vary across periods). At each settlement date t_i, the floating-leg payer pays interest at the realized forward rate $f^*_{[t_{i-1},t_i]}$ on the notional Q_{t_i}, while the fixed-leg payer pays interest at a constant rate R on the same notional.

Definition 5 (Par Swap Rate). The par swap rate R is the fixed rate such that the market value of the swap is zero at inception. Equivalently, R is determined by the condition that the present value of the fixed leg equals the present value of the floating leg: $\underbrace{\sum_{i=1}^{n} Q_{t_i}\, f^*_{[t_{i-1},t_i]}\, P_{t_i}}_{\text{PV of floating leg}} = \underbrace{\sum_{i=1}^{n} Q_{t_i}\, R\, P_{t_i}}_{\text{PV of fixed leg}}.$

Theorem 6 (General Swap Rate Formula). Under the no-arbitrage condition (3.1), the par swap rate is given by $\boxed{R = \frac{\sum_{i=1}^{n} Q_{t_i}\, f^*_{[t_{i-1},t_i]}\, P_{t_i}}{\sum_{i=1}^{n} Q_{t_i}\, P_{t_i}}}.$

Proof. Since is constant across all settlement periods, we factor it out of the right-hand side of (3.1):R

$$\sum_{i=1}^{n} Q_{t_i}\, f^*_{[t_{i-1},t_i]}\, P_{t_i} = R \sum_{i=1}^{n} Q_{t_i}\, P_{t_i}.$$

The denominator since all discount factors and notional amounts are strictly positive. Dividing both sides yields (3.2). □$\sum_{i=1}^{n} Q_{t_i}\, P_{t_i} > 0$

Remark 7. Formula (3.2) expresses as a weighted average of the forward rates , where the weights are proportional to . Intuitively, the par swap rate is the "average" forward rate, weighted by the present value of each notional exposure.(3.2)$R f^*_{[t_{i-1},t_i]} Q_{t_i} P_{t_i}$

4.0 Level Notional: Simplified Formula

In practice, most plain-vanilla interest rate swaps have a constant (level) notional amount. This common special case admits a remarkably elegant closed-form expression.

Theorem 8 (Level-Notional Swap Rate). If the notional amount is constant, i.e., $Q_{t_i} = Q$ for all $i = 1, \ldots, n$, then the par swap rate simplifies to $\boxed{R = \frac{1-P_{t_n}}{\sum_{i=1}^{n} P_{t_i}}.}$

Proof. When for all , the general formula (3.2) becomes$Q_{t_i} = Qi$(3.2)

$$R = \frac{\sum_{i=1}^{n} Q\, f^*_{[t_{i-1},t_i]}\, P_{t_i}}{\sum_{i=1}^{n} Q\, P_{t_i}} = \frac{\sum_{i=1}^{n} f^*_{[t_{i-1},t_i]}\, P_{t_i}}{\sum_{i=1}^{n} P_{t_i}}.$$

Applying Lemma 2.3 to the numerator:

$$\sum_{i=1}^{n} f^*_{[t_{i-1},t_i]}\, P_{t_i} = \sum_{i=1}^{n} \left(P_{t_{i-1}} - P_{t_i}\right).$$

This is a telescoping sum:

$$\begin{aligned}\sum_{i=1}^{n} \left(P_{t_{i-1}} - P_{t_i}\right) &= \left(P_{t_0} - P_{t_1}\right) + \left(P_{t_1} - P_{t_2}\right) + \cdots + \left(P_{t_{n-1}} - P_{t_n}\right) \\ &= P_{t_0} - P_{t_n} = 1 - P_{t_n},\end{aligned}$$

where we used . Substituting into the expression for yields (4.1). □ $P_{t_0} = P_0 = (1 + r_0)^0 = 1R$

Remark 9. Formula (4.1) has a natural economic interpretation. The numerator is the difference between \$1 today and the present value of \$1 at maturity. The denominator is the present value of an annuity paying \$1 per period. Thus, is the coupon rate of a par bond, which connects swap rates directly to bond pricing.(4.1)$1 - P_{t_n} \sum_{i=1}^{n} P_{t_i}\, R$

Corollary 10 (Par Bond Equivalence). The level-notional par swap rate R equals the coupon rate of a par-valued fixed-rate bond with the same payment dates and term structure.

Proof. A par bond with face value F and coupon rate c satisfies

$$F = \sum_{i=1}^{n} c\, F\, P_{t_i} + F\, P_{t_n},$$

which gives $1 = c \sum_{i=1}^{n} P_{t_i} + P_{t_n}$, hence $c = \left(1 - P_{t_n}\right) / \sum_{i=1}^{n} P_{t_i} = R$. □

5.0 Deferred-Start Swaps

A deferred-start swap (or forward-starting swap) is a swap that begins at some future date t_k $(k \geq 1)$ rather than at inception. Such instruments are useful for hedging anticipated future borrowing or for speculating on future interest rate movements.

Definition 11 (Deferred Par Swap Rate). Consider a swap with settlement dates $t_{k+1}, t_{k+2}, \ldots, t_n$ (i.e., the swap starts at time t_k and has its first payment at t_{k+1}). The deferred par swap rate R_d is the fixed rate satisfying $\sum_{i=k+1}^{n} Q_{t_i} \, f^*_{[t_{i-1},t_i]} \, P_{t_i} = R_d \sum_{i=k+1}^{n} Q_{t_i} \, P_{t_i}$.

Theorem 12 (Deferred Swap Rate — General Notional). The deferred par swap rate under general notional amounts is $R_d = \frac{\sum_{i=k+1}^{n} Q_{t_i} \, f^*_{[t_{i-1},t_i]} \, P_{t_i}}{\sum_{i=k+1}^{n} Q_{t_i} \, P_{t_i}}$.

Theorem 13 (Deferred Swap Rate — Level Notional). If $Q_{t_i} = Q$ for all $i = k+1, \ldots, n$, then $\boxed{R_d = \frac{P_{t_k} - P_{t_n}}{\sum_{i=k+1}^{n} P_{t_i}}}$.

Proof. Following the same telescoping argument as in the proof of Theorem 4.1:

$$\sum_{i=k+1}^{n} f^*_{[t_{i-1},t_i]} \, P_{t_i} = \sum_{i=k+1}^{n} \left(P_{t_{i-1}} - P_{t_i}\right) = P_{t_k} - P_{t_n}.$$

Dividing by yields (5.3). □$\sum_{i=k+1}^{n} P_{t_i}$

Remark 14. Theorem 4.1 is the special case of Theorem 5.3 with $k = 0$ (no deferral), since $P_{t_0} = P_0 = 1$.

6.0 Numerical Examples

In this section, we present a series of numerical examples that progressively illustrate the application of the formulas derived above.

6.1 Basic Application: Two-Year General Formula

Example 1. A bank borrows \$1,000,000 for two years at a variable interest rate. The current term structure consists of a one-year spot rate of $r_1 = 6\%$ and a two-year spot rate of $r_2 = 10\%$. Determine the par swap rate.

Solution. We compute the required quantities:

$$
\begin{aligned}
P_1 &= (1.06)^{-1} = 0.943396, \\
P_2 &= (1.10)^{-2} = 0.826446, \\
f^*_{[0,1]} &= r_1 = 0.06, \\
f^*_{[1,2]} &= \frac{(1.10)^2}{(1.06)^1} - 1 = \frac{1.21}{1.06} - 1 = 0.141509.
\end{aligned}
$$

Since the notional amount $Q = 1{,}000{,}000$ is constant, we apply the general formula (Theorem 3.1):

$$
\begin{aligned}
R &= \frac{(0.06)(0.943396) + (0.141509)(0.826446)}{0.943396 + 0.826446} \\
&= \frac{0.056604 + 0.116950}{1.769843} = \frac{0.173554}{1.769843} = 0.09806.
\end{aligned}
$$

Verification using the level-notional formula (Theorem 4.1):

$$
R = \frac{1 - P_2}{P_1 + P_2} = \frac{1 - 0.826446}{0.943396 + 0.826446} = \frac{0.173554}{1.769843} = 0.09806.
$$

The par swap rate is $R = \mathbf{9.806}\%$.

6.2 Application with Zero-Coupon Bond Prices

Example 2. Peter borrows $200,000 to be repaid at the end of five years at a floating rate, and wishes to convert to a fixed rate via a swap. Zero-coupon bond prices (face value $1) are given in Table 1.

Table 1: Prices of Zero-Coupon Bonds (Face Value $1)

Maturity (years)	**Price P_t**
1	0.8500
2	0.8400
3	0.7900
4	0.7700
5	0.7200

Solution. Since the notional is level, we apply Theorem 4.1:

$$
\begin{aligned}
R &= \frac{1 - P_5}{\sum_{i=1}^{5} P_i} = \frac{1 - 0.72}{0.85 + 0.84 + 0.79 + 0.77 + 0.72} = \frac{0.28}{3.97} = 0.07053 \\
&= \mathbf{7.053}\%.
\end{aligned}
$$

6.3 Comparison of General and Level-Notional Formulas

For Examples Example 6.3–Example 6.6, we use the term structure in Table 2 and the derived quantities in Table 3.

Table 2: Term Structure of Spot Interest Rates

Time t (years)	Spot Rate r_t
1	4.00%
2	5.00%
3	5.75%
4	6.25%
5	6.50%

Table 3: Derived Discount Factors and One-Period Forward Rates

t	r_t	P_t = (1+r_t)^(-t)	f*[t-1,t]	f*[t-1,t] · P_t
1	4.00%	0.961538	0.040000	0.038462
2	5.00%	0.907029	0.060096	0.054509
3	5.75%	0.845588	0.072661	0.061441
4	6.25%	0.784665	0.077642	0.060923
5	6.50%	0.729881	0.075059	0.054784
Σ	—	**4.228702**	—	**0.270119**

Example 3 (Two-Year Level-Notional Swap). Jacque has a variable-rate loan of $5,000 for two years with annual resets. Determine the fixed swap rate using both formulas.

Solution.

General formula (Theorem 3.1):

$$R = \frac{(5000)(0.04)(0.961538) + (5000)(0.060096)(0.907029)}{(5000)(0.961538) + (5000)(0.907029)}$$
$$= \frac{192.308 + 272.545}{9{,}338.84} = 0.049755.$$

Level-notional formula (Theorem 4.1):

$$R = \frac{1 - P_2}{P_1 + P_2} = \frac{1 - 0.907029}{0.961538 + 0.907029} = \frac{0.092971}{1.868568} = 0.049755.$$

Both formulas yield $R = \mathbf{4.976}\%$, confirming the equivalence established in Section 4.

Example 4 (Five-Year Level-Notional Swap). Miles Manufacturing Corporation enters into a five-year swap with a constant notional of \$300,000 and annual settlement periods. Determine the par swap rate.

Solution. Applying Theorem 4.1 with data from Table 3:

$$R = \frac{1 - P_5}{\sum_{i=1}^{5} P_i} = \frac{1 - 0.729881}{4.228702} = \frac{0.270119}{4.228702} = \mathbf{6.388}\%.$$

Example 5 (Variable Notional Swap). James and Associates has a line of credit: \$400,000 in year 1, \$600,000 in year 2, and \$1,000,000 in year 3 (cumulative). James enters into a three-year swap matching these notional amounts. Determine the par swap rate.

Solution. Since the notional amounts vary, the general formula (Theorem 3.1) must be used:

$$\begin{aligned} \text{Numerator} &= (400{,}000)(0.04)(0.961538) + (600{,}000)(0.060096)(0.907029) \\ &\quad + (1{,}000{,}000)(0.072661)(0.845588) \\ &= 15{,}384.62 + 32{,}705.44 + 61{,}441.32 = 109{,}531.37, \\ \text{Denominator} &= (400{,}000)(0.961538) + (600{,}000)(0.907029) + (1{,}000{,}000)(0.845588) \\ &= 384{,}615.38 + 544{,}217.69 + 845{,}588.12 = 1{,}774{,}421.19. \end{aligned}$$

Thus, $R = 109{,}531.37 \,/\, 1{,}774{,}421.19 = \mathbf{6.173}\%$.

Example 6 (Deferred-Start Swap). Shyu & Salisbury Actuarial Consultants enter into a deferred interest rate swap with a level notional of \$125,000. The swap covers years 3–5 of a five-year term (no settlement in years 1–2). Determine the deferred par swap rate.

Solution. This is a deferred swap with $k = 2$, so we apply Theorem 5.3:

$$R_d = \frac{P_2 - P_5}{\sum_{i=3}^{5} P_i} = \frac{0.907029 - 0.729881}{0.845588 + 0.784665 + 0.729881} = \frac{0.177148}{2.360134}$$
$$= \mathbf{7.506}\%.$$

Verification via Theorem 5.2:

$$R_d = \frac{(0.072661)(0.845588) + (0.077642)(0.784665) + (0.075059)(0.729881)}{0.845588 + 0.784665 + 0.729881}$$
$$= \frac{0.061441 + 0.060923 + 0.054784}{2.360134} = \frac{0.177148}{2.360134} = 7.506\%.$$

Both formulas agree, confirming the result.

7.0 Sensitivity Analysis

To illustrate the economic content of the swap rate formulas, we analyze the sensitivity of the par swap rate to changes in the term structure.

7.1 Parallel Shifts in the Yield Curve

We consider the baseline upward-sloping term structure of Table 2 and examine the effect of parallel shifts of $\Delta \in \{-200, -100, 0, +100, +200\}$ basis points on the five-year level-notional swap rate.

Table 4: Five-Year Par Swap Rate Under Parallel Yield Curve Shifts

Shift (bps)	r_1	r_2	r_3	r_4	r_5	Swap Rate R
-200	2.00%	3.00%	3.75%	4.25%	4.50%	4.422%
-100	3.00%	4.00%	4.75%	5.25%	5.50%	5.405%
0	4.00%	5.00%	5.75%	6.25%	6.50%	6.388%
$+100$	5.00%	6.00%	6.75%	7.25%	7.50%	7.371%
$+200$	6.00%	7.00%	7.75%	8.25%	8.50%	8.354%

The swap rate increases approximately linearly with parallel shifts, though a slight convexity effect is present due to the nonlinear discount factor function $(1 + r)^{-t}$.

7.2 Yield Curve Shape Effects

Table 5 compares swap rates across three distinct yield curve shapes. Under a flat yield curve, the swap rate equals the common spot rate, as expected from (4.1) and Corollary 10. An upward-sloping curve produces a swap rate above the short end but below the long end, reflecting the weighted-average structure of the formula. An inverted curve produces a lower swap rate, capturing declining expected future rates.

Table 5: Five-Year Par Swap Rate Under Different Yield Curve Shapes

Yield Curve Shape	**r_1**	**r_2**	**r_3**	**r_4**	**r_5**	**Swap Rate R**
Flat	5.00%	5.00%	5.00%	5.00%	5.00%	5.000%
Upward-sloping	4.00%	5.00%	5.75%	6.25%	6.50%	6.388%
Inverted	6.50%	6.25%	5.75%	5.00%	4.00%	4.127%

8.0 Conclusion

This paper has presented a rigorous discrete-time framework for the valuation of interest rate swaps. Our main contributions are:

1.A self-contained derivation of the general par swap rate formula (Theorem 3.1) under time-varying notional amounts, grounded in the no-arbitrage principle.

2.A proof that the level-notional formula (Theorem 4.1) follows from the general formula via the telescoping property (Lemma 2.3), with the further equivalence to par bond coupon rates established in Corollary 4.3.

3.Extension to deferred-start swaps with both general (Theorem 5.2) and level-notional (Theorem 5.3) closed-form formulas.

4.Comprehensive numerical illustrations and a sensitivity analysis demonstrating the relationship between yield curve shape and par swap rates.

Several natural extensions merit further investigation:

Net swap payments. In practice, counterparties exchange only the net difference. At each settlement date t_i, the net payment from the fixed-rate payer to the floating-rate payer is $Q_{t_i}\left(R - f^*_{[t_{i-1},t_i]}\right)$, which may be positive or negative depending on the realized forward rate.

Market value of swaps post-inception. At inception, the swap's market value is zero by construction. As time passes and the yield curve evolves, the swap acquires a non-zero market value. Denoting the updated yield curve at time $s > 0$ by $\{\tilde{r}_t\}$, the market value to the fixed-rate payer is

$$V_s = \sum_{i:\, t_i > s} Q_{t_i}\left(\tilde{f}^*_{[t_{i-1},t_i]} - R\right)\tilde{P}_{t_i - s},$$

where $\tilde{P}$ and $\tilde{f}^*$ are computed from the updated term structure.

Continuous-time extensions. The discrete-time framework naturally extends to continuous-time models. In the Heath–Jarrow–Morton framework Heath et al. (1992), the instantaneous forward rate $f(t,T)$ satisfies a stochastic differential equation, and swap rates are computed as expectations under the appropriate forward measure. The LIBOR market model Brace et al. (1997) and Miltersen et al. (1997) provides an alternative approach that directly models discrete forward rates and is particularly suited for pricing interest rate derivatives.

Credit risk and CVA. The framework presented here assumes no counterparty default risk. In practice, credit valuation adjustment (CVA) is added to account for the possibility that one counterparty may default before the swap matures Hull (2012).

Multi-curve framework. Since the 2007–2008 financial crisis, the market has adopted a multi-curve framework where discounting uses overnight index swap (OIS) rates while forward rates are derived from the relevant IBOR curve Henrard (2014). This introduces basis spreads and additional complexity into swap valuation that the single-curve framework does not capture.

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